A NOTE ON SELF-DUAL REPRESENTATIONS OF $\mathrm{Sp}(4, F)$

KUMAR BALASUBRAMANIAN

ABSTRACT. Let F be a non-Archimedean local field of characteristic 0 and $G = \mathrm{Sp}(4, F)$. Let (π, W) be an irreducible smooth self-dual representation of G. The space W of π admits a non-degenerate G-invariant bilinear form $(\ ,\ )$ which is unique up to scaling. It can be shown that $(\ ,\ )$ is either symmetric or skew-symmetric and we set $\varepsilon(\pi) = \pm 1$ accordingly. In this paper, we show that $\varepsilon(\pi) = 1$ when π is an Iwahori-spherical representation of G.

1. Introduction

Let G be a group and (π, V) be an irreducible complex representation of G. Suppose that $\pi \simeq \pi^\vee$ ($\pi^\vee$ is the dual or contragredient representation). In the presence of Schur's lemma, it can be shown that there exists a non-degenerate G-invariant bilinear form on V which is unique up to scalars, and consequently is either symmetric or skew-symmetric. Accordingly, we set

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric,} \\ -1 & \text{if the form is skew-symmetric,} \end{cases}$$

which we call the sign of π. In this paper, we study this sign for a certain class of representations of $\mathrm{Sp}(4, F)$. To be more precise, we show that $\varepsilon(\pi) = 1$, when π is an Iwahori-spherical representation of $G = \mathrm{Sp}(4, F)$.

The sign $\varepsilon(\pi)$ has been well studied for connected compact Lie groups and certain classes of finite groups of Lie type. If G is a connected compact Lie group, it is known that the sign can be computed using the dominant weight attached to the representation π (see [4] pg. 261-264). For certain finite classical groups, computing the sign involves tedious conjugacy class computations. We refer to the following paper of Gow ([7]) where the sign is studied for such groups. In [12], Prasad introduced an idea to compute the sign for a certain class of representations of finite groups of Lie type. He used this idea to determine the sign for many classical groups of Lie type, avoiding difficult computations. In recent times, there has been a lot of progress in studying these signs in the setting of reductive p-adic groups. In [13], Prasad extended the results of [12] to the case of reductive p-adic groups and computed the sign of certain classical groups. The disadvantage of his method is that it works only for representations admitting a Whittaker model. In [15], Roche and Spallone discuss the relation between twisted sign (see section 1 in [15]) and the ordinary sign and describe a way of studying the ordinary sign using the twisted sign. More recently in [14], Prasad and Ramakrishnan have looked at signs of irreducible self-dual discrete series representations of $\mathrm{GL}_n(D)$, for D a finite

Supported by DST-SERB Grant: YSS/2014/000806.

dimensional p-adic division algebra, and have proved a remarkable formula that relates the signs of these representations and the signs of their Langlands parameters.

In this paper, we compute $\varepsilon(\pi)$ for any irreducible smooth self-dual representation of $\mathrm{Sp}(4, F)$ with non-trivial vectors fixed under an Iwahori subgroup. To be more precise, we prove the following

Theorem 1.1 (Main Theorem)**.** *Let $G = \mathrm{Sp}(4, F)$ and (π, W) be an irreducible smooth self-dual representation of G with non-trivial vectors fixed under an Iwahori subgroup. Then $\varepsilon(\pi) = 1$.*

We briefly explain the key ideas of the proof. We first consider the case when π is a square-integrable representation of G. We prove that such representations are generic. When π is a generic representation of G, it is well known from the work of Prasad (proposition 2 in [12])) that the sign is given by $\omega_\pi(-1)$, where ω_π is the central character of π. Using this along with the fact that π is Iwahori-spherical, it follows immediately that $\varepsilon(\pi) = 1$.

In the case when π is not generic, we consider the following two cases:

a) π is not tempered: Here we use the results of Roche and Spallone ([15]) and reduce the problem to computing the twisted sign (explained later) of a certain tempered representation of a Levi subgroup of G. Using the fact that the Levi subgroup of G is of a certain type, allows us to apply some known results on Iwahori-spherical representations to compute the sign.

b) π is tempered: Here we compute the sign by reducing to the case of studying the twisted sign of a certain discrete series representation of a Levi subgroup of G.

Remark 1.2. The fact that square-integrable, Iwahori-spherical representations of $\mathrm{Gsp}(4, F)$ are generic is crucial in proving our main result. Suppose we are able to compute the signs for square-integrable, Iwahori-spherical representations of $\mathrm{Sp}(n, F)$ (for $n > 4$), the methods of this paper combined with some inductive arguments can be used to determine the signs for $\mathrm{Sp}(n, F)$ (for $n > 4$). We hope to address this problem at a later time.

The paper is organized as follows. In section 2, we introduce the notion of twisted and ordinary signs attached to the representation π. In section 3, we recall some results which we need. In the section 4 we prove the main theorem.

2. Some preliminaries on signs

In this section, we discuss the notion of twisted and ordinary signs associated to representations.

Let F be a non-Archimedean local field and G be the group of F-points of a connected reductive algebraic group. Let (π, W) be a smooth irreducible complex representation of G. We write $(\pi^\vee, W^\vee)$ for the smooth dual or contragredient of (π, W) and $\langle\, ,\, \rangle$ for the canonical non-degenerate G-invariant pairing on $W \times W^\vee$ (given by evaluation). Let θ be a continuous automorphism of G of order at most two. Let (π^θ, W) be the θ-twist of π defined by

$$\pi^\theta(g)w = \pi(\theta(g))w.$$

Suppose that $\pi^\theta \simeq \pi^\vee$. Let $s : (\pi^\theta, W) \to (\pi^\vee, W^\vee)$ be an isomorphism. The map s can be used to define a bilinear form on W as follows:

$$(w_1, w_2) = \langle w_1, s(w_2)\rangle, \quad \forall\, w_1, w_2 \in W.$$

It is easy to see that $(\, ,\,)$ is a non-degenerate form on W that satisfies the following invariance property

$$(\pi(g)w_1, \pi^\theta(g)w_2) = (w_1, w_2), \quad \forall\, w_1, w_2 \in W. \tag{2.1}$$

Let $(\, ,\,)_*$ be a new bilinear form on W defined by

$$(w_1, w_2)_* = (w_2, w_1)$$

Clearly, this form is again non-degenerate and G-invariant in the sense of (2.1). It follows from Schur's Lemma that

$$(w_1, w_2)_* = c(w_1, w_2)$$

for some non-zero scalar c. A simple computation shows that $c \in \{\pm 1\}$. Indeed,

$$(w_1, w_2) = (w_2, w_1)_* = c(w_2, w_1) = c(w_1, w_2)_* = c^2(w_1, w_2).$$

We set $c = \varepsilon_\theta(\pi)$ and call it the twisted sign of π. It clearly depends only on the equivalence class of π. If θ is the trivial automorphism of G, we simply write $\varepsilon(\pi)$ instead of $\varepsilon_1(\pi)$ and call it the ordinary sign. In sum, the form $(\, ,\,)$ is symmetric or skew-symmetric and the sign $\varepsilon_\theta(\pi)$ determines its type.

2.1. Let θ be an automorphism of G of order at most 2 and suppose that $\pi^\theta \simeq \pi^\vee$. Consider the automorphism θ' of G defined by

$$\theta' = \mathrm{Int}(h) \circ \theta$$

for $h \in G$, where $\mathrm{Int}(h)$ denotes the inner automorphism $g \to hgh^{-1}$ of G. In this situation, it is clear that $\pi^{\theta'} \simeq \pi^\vee$. Suppose $h \in G$ is chosen such that θ' has order at most two. It is clear that $h\theta(h)$ is a central element and $h\theta(h) = \theta(h)h$. A simple computation shows that

$$\varepsilon_{\theta'}(\pi) = \varepsilon_\theta(\pi)\omega_\pi(\theta(h)h) \tag{2.2}$$

where ω_π is the central character. For specific details of this computation, we refer the reader to 1.2.1 in [15].

3. Some results we need

In this section, we recall some results used in the proof of the main result of this paper.

3.1. Prasad's Theorem. Throughout this section we write F for a non-Archimedean local field of characteristic 0. In [13], Prasad shows that for an irreducible smooth self-dual generic representation of $\mathrm{Sp}(n, F)$, the sign $\varepsilon(\pi)$ is determined by the value of the central character ω_π of π. To be more precise, he proves the following

Theorem 3.1 (Prasad)**.** *π be an irreducible smooth self-dual generic representation of $\mathrm{Sp}(n, F)$. Then $\varepsilon(\pi) = \omega_\pi(-1)$.*

We refer the reader to [12] where the result is proved for $\mathrm{Sp}(n, F_q)$, where F_q is a finite field with q elements. Similar ideas along with the idea of compact approximation of Whittaker models ([17], section III, pg. 155), can be used to prove the above result when the finite field F_q is replaced with a non-Archimedean local field F.

3.2. Waldspurger's Theorem. In this section, we recall a result of Waldspurger which we need in the proof of the main theorem.

3.2.1. Throughout this section, we let F be a non-Archimedean local field of characteristic $\neq 2$ and W be a finite dimensional vector space over F. We let $\langle\, , \rangle$ to be a non-degenerate symmetric or skew-symmetric form on W. We take

$$G = \{g \in \mathrm{GL}(W) \mid \langle gw, gw' \rangle = \langle w, w' \rangle\}.$$

For $x \in \mathrm{GL}(W)$ such that $xGx^{-1} = G$ and (π, V) a representation of G, we let π^x denote the representation of G defined by conjugation (i.e., $\pi^x(g) = \pi(xgx^{-1})$).

We recall the statement of Waldspurger's theorem below and refer the reader to Chapter 4.II.1 in [11] for a proof.

Theorem 3.2 (Waldspurger)**.** *Let π be an irreducible admissible representation of G and $\pi^\vee$ be the smooth-dual or contragredient of π. Let $x \in \mathrm{GL}(W)$ be such that $\langle xw, xw' \rangle = \langle w', w \rangle, \forall\, w, w' \in W$. Then $\pi^x \simeq \pi^\vee$.*

3.3. Harish-Chandra's Theorem. Throughout this section, we take $G = G(F)$, $T = T(F)$ and write $\mathcal{W} = W(G, T)$ for the Weyl group of G with respect to the maximal split torus T. For H a subgroup of G, $s \in \mathcal{W}$, and (σ, V) a representation of H, we let ${}^s\sigma(x) = \sigma(s^{-1}xs)$. We let $\mathrm{ind}_P^G \sigma$ for the normalized parabolically induced representation from P to G.

Theorem 3.3 (Harish-Chandra)**.** *Let π be an irreducible admissible tempered representation of G. Then:*

- (i) *There exists a parabolic subgroup $P = MU$ and an irreducible admissible square-integrable representation σ of M such that π is a sub-quotient of $\mathrm{ind}_P^G \sigma$.*
- (ii) *If $(P = MU, \sigma)$, $(P' = M'U', \sigma')$ are two representations of G satisfying (i) above, then there exists $s \in \mathcal{W}$ such that $sMs^{-1} = M'$ and ${}^s\sigma = \sigma'$.*

We refer the reader to III.4.1 in [20] for the proof of the above result.

3.4. Reduction to Tempered case. Throughout this section, we use the same notation and terminology as in [15]. In [15], Roche and Spallone reduce the problem of computing the θ-twisted sign to the case of tempered representations. We briefly recall their method below. For further details, we refer the reader to sections §3, §4 of [15].

Let θ be an involutory automorphism of G and suppose that $\pi^\theta \simeq \pi^\vee$. Let (P, τ, ν) be the triple associated to π via the Langlands' classification. Suppose that P has Levi decomposition $P = MN$. Under certain assumptions on the involution θ, they apply Casselman's pairing to show that $\varepsilon_\theta(\pi) = \varepsilon_\theta(\pi_N)$, where π_N is the Jacquet module of π. Using $\pi^\theta \simeq \pi^\vee$ and the fact that τ occurs with multiplicity one as a composition factor of π_N, they prove the following

Theorem 3.4 (Roche-Spallone). *Let π be an irreducible smooth representation of G such that $\pi^\theta \simeq \pi^\vee$. Suppose the Langlands' classification attaches the triple (P, τ, ν) to π. Then $\tau^\theta \simeq \tau^\vee$ and $\varepsilon_\theta(\pi) = \varepsilon_\theta(\tau)$.*

Remark 3.5. In the above theorem, for τ^θ to make sense, we need that θ preserves M, i.e., $\theta(M) = M$. This will follow from the assumptions on the involution θ.

3.5. Restriction of representations. We recall some results about restricting an irreducible representation to a subgroup. These results hold when G is a locally compact totally disconnected group and H is an open normal subgroup of G such that G/H is (finite) abelian. For a more detailed account, we refer the reader to [5] (Lemmas 2.1 and 2.3).

Theorem 3.6 (Gelbart-Knapp). *Let π be an irreducible admissible representation of G. Suppose that G/H is (finite) abelian. Then*

(i) *$\pi|_H$ is a finite direct sum of irreducible admissible representations of H.*
(ii) *When the irreducible constituents of $\pi|_H$ are grouped according to their equivalence classes as*

$$\pi|_H = \bigoplus_{i=1}^{M} m_i \pi_i$$

with the π_i irreducible and inequivalent, the integers m_i are equal.

Theorem 3.7 (Gelbart-Knapp). *Let G be a locally compact totally disconnected group and H be an open normal subgroup of G such that G/H is (finite) abelian, and let π be an irreducible admissible representation of H. Then*

(i) *There exists an irreducible admissible representation $\tilde{\pi}$ of G such that $\tilde{\pi}|_H$ contains π as a constituent.*
(ii) *Suppose $\tilde{\pi}$ and $\tilde{\pi}'$ are irreducible admissible representations of G whose restrictions to H are multiplicity free and contain π. Then $\tilde{\pi}|_H$ and $\tilde{\pi}'|_H$ are equivalent and $\tilde{\pi}$ is equivalent with $\tilde{\pi}' \otimes \chi$ for some character χ of G that is trivial on H.*

4. Main Theorem

Throughout this section, we set $G = \mathrm{Sp}(4, F)$ and write I for an Iwahori subgroup in G. We let π denote an irreducible smooth self-dual Iwahori spherical representation of G. We prove that $\varepsilon(\pi) = 1$.

We consider the following cases to prove the main theorem:

a) π is generic.
b) π is square-integrable.
c) π is not generic.

4.1. π is a generic representation. When π is a generic representation of G, it is well known that $\varepsilon(\pi) = \omega_\pi(-1)$ (see §7, proposition 2 in [12]). Since π is Iwahori spherical, it is easy to see that $\omega_\pi(-1) = 1$ and the result follows. Indeed, for $0 \neq v_0 \in \pi^I$, we have

$$v_0 = \pi(-1)v_0 = \omega_\pi(-1)v_0.$$

4.2. π is square-integrable. Throughout this section, we let $G = \mathrm{Sp}(4, F)$ and $\tilde{G} = \mathrm{Gsp}(4, F)$. We write T (respectively $\tilde{T}$) for a maximal F-split torus in G (respectively $\tilde{G}$). We write I (respectively $\tilde{I}$) for an Iwahori subgroup in G (respectively $\tilde{G}$). For $g \in \tilde{G}$, we let $\lambda_g \in F^\times$ denote the multiplier of g. It is easy to see that the map $g \to \lambda_g$ defines a homomorphism between $\tilde{G}$ and $F^\times$. We denote it by λ.

Let (π, W) be an irreducible smooth self-dual square-integrable representation of G with non-trivial vectors fixed under an Iwahori subgroup I. We know that there exists an irreducible smooth square-integrable representation $(\tilde{\pi}, V)$ of $\tilde{G}$ such that the restriction of $\tilde{\pi}$ to G contains the representation π. We can show that the representation $\tilde{\pi}$ can be chosen in such a way that $\tilde{\pi}^{\tilde{I}} \neq 0$. We do this by choosing a character τ_1 of $\tilde{I}$ which is trivial on I and extending it to a character $\tilde{\tau}$ of $\tilde{G}$ which is trivial on G and considering the representation $\tilde{\pi}\tilde{\tau}^{-1}$ (see Lemma 4.11, Theorem 4.12 in [2]). To simplify notation, we denote $\tilde{\pi}\tilde{\tau}^{-1}$ again as $\tilde{\pi}$. By construction, the character $\tilde{\tau}$ is a unitary character of $\tilde{G}$ and it follows that the modified representation $\tilde{\pi}$ is a square-integrable representation which is also Iwahori-spherical and contains the representation π on restriction to G. Since $\tilde{\pi}$ is an Iwahori-spherical and a square-integrable representation, it is generic (see [18], [19]). It follows that π is also generic and by the previous case, we have $\varepsilon(\pi) = 1$.

4.3. π is not a generic representation. We now consider the case when π is not a generic representation of G. In this case, we prove the result by considering further classifications of the representation π. To be more precise, we consider the following cases separately

(a) π is not tempered.
(b) π is tempered.

Before we proceed further, we set up some notation and record a few observations. For vectors $v = (v_1, v_2, v_3, v_4), u = (u_1, u_2, u_3, u_4) \in F^4$, we let

$$\langle v, u\rangle' = v_1u_4 + v_2u_3 - v_3u_2 - v_4u_1$$

to be the skew-symmetric bilinear form used to define the symplectic group and let $w_0 \in \mathrm{GL}(4, F)$ to be the matrix with anti-diagonal entries 1. It is clear that w_0

normalizes G and satisfies $w_0^2 = 1$. Also for $g \in G$, we have

$$\begin{aligned}\langle w_0 g w_0^{-1} v, w_0 g w_0^{-1} u\rangle' &= \langle g w_0^{-1} u, g w_0^{-1} v\rangle' \\ &= \langle w_0^{-1} u, w_0^{-1} v\rangle' \\ &= \langle v, u\rangle'.\end{aligned}$$

By theorem 3.2 above, it follows that $\pi^\theta \simeq \pi^\vee$, where $\theta(g) = w_0 g w_0^{-1}$. For $g \in \tilde{G} = \mathrm{Gsp}(4, F)$, we let $\iota(g) = \lambda_g^{-1} w_0 g w_0^{-1}$, where λ_g is the multiplier of g. It is easy to see that ι is a continuous automorphism of $\tilde{G}$ of order two and the restriction of ι to G is θ. Let $\tilde{\pi}$ be an irreducible smooth representation of $\tilde{G}$ such that the restriction of $\tilde{\pi}$ to G contains the representation π with multiplicity one (see [1]). It can be shown that $\tilde{\pi}^\iota \simeq \tilde{\pi}^\vee$ (we refer the reader to Theorem B in [16] for a proof) and thus $\varepsilon_\iota(\tilde{\pi})$ makes sense. Since $\iota|_G = \theta$, and the restriction of the bilinear form on $\tilde{\pi}$ satisfying

$$[\tilde{\pi}(g)(v), \tilde{\pi}^\iota(g)(w)] = [v, w] \tag{4.1}$$

to π is non-degenerate; we have

$$\varepsilon_\iota(\tilde{\pi}) = \varepsilon_\theta(\pi). \tag{4.2}$$

Consider the representation $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \omega_{\tilde{\pi}}^{-1}$ (Here $\omega_{\tilde{\pi}}$ is the character obtained by composing the central character $\omega_{\tilde{\pi}}$ with the map $\lambda : \tilde{G} \to F^\times$). Since $\pi \simeq \pi^\vee$, it follows that the restriction of $\tilde{\pi}^\vee$ to G also contains π with multiplicity one. The bilinear form $(\ ,\ )$ on $\tilde{\pi}$ satisfying

$$(\tilde{\pi}(g)v, \tilde{\pi}(g)w) = \omega_{\tilde{\pi}}(\lambda_g)(v, w)$$

restricts to a non-degenerate G-invariant bilinear form on π and it follows that

$$\varepsilon(\tilde{\pi}) = \varepsilon(\pi). \tag{4.3}$$

We use the above results to establish a relation between the twisted sign $\varepsilon_\theta(\pi)$ and the ordinary sign $\varepsilon(\pi)$. We record the result in the following lemma.

Lemma 4.1. $\varepsilon_\theta(\pi) = \varepsilon(\pi)\omega_\pi(-1)$.

Proof. Consider the non-degenerate bilinear form on $\tilde{\pi}$ defined by

$$[v_1, v_2]' = (v_1, \tilde{\pi}(w_0^{-1})v_2).$$

It is clearly $\tilde{G}$ invariant in the sense of (4.1). Indeed,

$$\begin{aligned}[\tilde{\pi}(g)v_1, \tilde{\pi}^\iota(g)v_2]' &= (\tilde{\pi}(g)v_1, \tilde{\pi}(w_0^{-1})\tilde{\pi}^\iota(g)v_2) \\ &= (\tilde{\pi}(g)v_1, \tilde{\pi}(w_0^{-1})\tilde{\pi}(\lambda_g^{-1} w_0 g w_0^{-1})v_2) \\ &= (\tilde{\pi}(g)v_1, \tilde{\pi}(w_0^{-1}\lambda_g^{-1} w_0 g w_0^{-1})v_2) \\ &= (\tilde{\pi}(g)v_1, \tilde{\pi}(\lambda_g^{-1})\tilde{\pi}(g w_0^{-1})v_2) \\ &= \omega_{\tilde{\pi}}(\lambda_g^{-1})(\tilde{\pi}(g)v_1, \tilde{\pi}(g w_0^{-1})v_2) \\ &= \omega_{\tilde{\pi}}(\lambda_g^{-1})\omega_{\tilde{\pi}}(\lambda_g)(v_1, \tilde{\pi}(w_0^{-1})v_2) \\ &= [v_1, v_2]'.\end{aligned}$$

It follows that

$$[v_1, v_2]' = \varepsilon_\iota(\tilde{\pi})[v_2, v_1]' = \varepsilon(\tilde{\pi})\omega_{\tilde{\pi}}(-1).$$

Indeed,

$$\begin{aligned}
[v_1, v_2]' &= \varepsilon_\iota(\tilde{\pi})[v_2, v_1]' \\
&= (v_1, \tilde{\pi}(w_0^{-1})v_2) \\
&= \varepsilon(\tilde{\pi})(\tilde{\pi}(w_0^{-1})v_2, v_1) \\
&= \varepsilon(\tilde{\pi})\omega_{\tilde{\pi}}(\lambda_{w_0}^{-1})(v_2, \tilde{\pi}(w_0)v_1) \\
&= \varepsilon(\tilde{\pi})\omega_{\tilde{\pi}}(-1)(v_2, \tilde{\pi}(w_0^{-1})v_1) \\
&= \varepsilon(\tilde{\pi})\omega_{\tilde{\pi}}(-1)[v_2, v_1]'.
\end{aligned}$$

The result now follows from (4.2) and (4.3).

□

Since π has non-trivial Iwahori fixed vectors, it follows that $\omega_\pi(-1)$ is trivial and hence

$$\varepsilon_\theta(\pi) = \varepsilon(\pi).$$

We now focus on computing the twisted sign attached to π.

4.3.1. *π is not tempered.* Let (P, τ, ν) be the triple associated to π via the Langlands' classification. We let M and N denote the Levi component and the unipotent radical of the parabolic subgroup P. Before we proceed further, we observe that the automorphism θ satisfies the hypotheses needed in order to apply Casselman's pairing as stated in §3 of [15]. To be more precise, we have

Lemma 4.2. *The involution θ satisfies the following conditions*

(i) *θ is an automorphism of G as an algebraic group.*
(ii) *θ preserves T so that $\theta|_T$ is an involutory automorphism of the F-split torus T and*
(iii) *θ maps N to the M-opposite $\bar{N}$.*

Proof. Let T be the standard maximal split torus in G and P be the standard parabolic subgroups of G consisting of suitable block matrices. A simple computation shows that the above properties are satisfied by θ. □

From lemma 4.2 and theorem 3.4, it follows that

$$\varepsilon_\theta(\pi) = \varepsilon_\theta(\tau). \tag{4.4}$$

Throughout we write $I_M = I \cap M$. Before we continue, we observe that τ has non-trivial I_M fixed vectors. We record the result in the following

Lemma 4.3. *The representation τ has non-trivial I_M fixed vectors.*

Proof. Since $\pi \hookrightarrow \mathrm{ind}_P^G(\tau\nu)$ and $\pi^I \neq 0$, it follows that $(\pi_N)^{I_M} \neq 0$. Since $\tau\nu$ occurs as a composition factor of π_N, it follows that $(\tau\nu)^{I_M} \neq 0$ (refer Lemma 4.7 and Lemma 4.8 in [3]). Since I_M is compact and ν is a (continuous) character of M taking positive real values, we have $\nu|_{I_M} = 1$ and $\tau^{I_M} \neq 0$. □

From (4.4), it is enough to compute the sign $\varepsilon_\theta(\tau)$ where τ is an irreducible tempered representation of M with $\tau^{I_M} \neq 0$. Since M is a Levi subgroup of G, we have the following possibilities for M:

(a) $M \simeq \mathrm{GL}(1,F) \times \mathrm{GL}(1,F)$,
(b) $M \simeq \mathrm{GL}(2,F)$,
(c) $M \simeq \mathrm{SL}(2,F) \times \mathrm{GL}(1,F)$.

In all these cases, we show that $\varepsilon_\theta(\tau) = 1$.

For cases (a) and (b), considering θ as an automorphism of $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F)$ and $\mathrm{GL}(2,F)$, it is easy to see that $\theta(g) =^\top g^{-1}$ where $^\top$ denotes the transpose. Now using $\tau^\theta \simeq \tau^\vee$ (as a representation of either $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F)$ or $\mathrm{GL}(2,F)$), and τ is tempered (hence generic) (see [10], [9]), it follows that $\varepsilon_\theta(\tau) = 1$ (§5, Theorem 1 in [15]).

We now consider the case when $M \simeq \mathrm{SL}(2,F) \times \mathrm{GL}(1,F)$. Since τ is an irreducible tempered representation of M, $\tau \simeq \tau_1 \otimes \tau_2$ where τ_1, τ_2 are irreducible tempered representations of $\mathrm{SL}(2,F)$ and $\mathrm{GL}(1,F)$ respectively. Since $\tau^\theta \simeq \tau^\vee$, we have

$$(\tau_1 \otimes \tau_2)^\theta \simeq \tau_1^\theta \otimes \tau_2^\theta \simeq (\tau_1 \otimes \tau_2)^\vee \simeq \tau_1^\vee \otimes \tau_2^\vee.$$

It is clear that $\tau_1^\theta \simeq \tau_1^\vee$ (here θ acts like the involution $\theta(g) = xgx^{-1}$, where x is the matrix with anti- diagonal entries 1) and $\tau_2^\theta \simeq \tau_2^\vee$ (here θ acts like the involution $\theta(g) =^\top g^{-1}$). It is also easy to see that $I_M = I \cap M \simeq I_{\mathrm{SL}(2,F)} \times I_{\mathrm{GL}(1,F)}$. Since $\tau^{I_M} \neq 0$, it follows that $\tau_1^{I_{\mathrm{SL}(2,F)}} \neq 0$ and $\tau_2^{I_{\mathrm{GL}(1,F)}} \neq 0$.

Clearly $\varepsilon_\theta(\tau_2) = 1$ (§5, Theorem 1 in [15]). Now consider the representation τ_1. Since τ_1 is a tempered representation of $\mathrm{SL}(2,F)$, it is generic (follows from Lemma 4.3 in [2]). We know that there exists an irreducible smooth representation $\tilde{\tau}_1$ of $\mathrm{GL}(2,F)$ such that $\tilde{\tau}_1|_{\mathrm{SL}(2,F)} \supset \tau_1$ with multiplicity one. We can in fact, choose $\tilde{\tau}_1$ such that $\tilde{\tau}_1^{I_{\mathrm{GL}(2,F)}} \neq 0$ (see Lemma 4.11 and Theorem 4.12 in [2]). It is clear that $(\tilde{\tau}_1^\theta)^\vee$ also contains the representation τ_1 with multiplicity one. It follows that

$$(\tilde{\tau}_1^\theta)^\vee \simeq \tilde{\tau}_1 \otimes \chi, \tag{4.5}$$

where χ is a character of $\mathrm{GL}(2,F)$ such that $\chi|_{\mathrm{SL}(2,F)} = 1$ (see Lemmas 2.1 and 2.3 in [5]). Using (4.5), it is easy to see that the space of $\tilde{\tau}_1$ admits a non-degenerate bilinear form $[\,,\,]$ satisfying

$$[\tilde{\tau}_1(g)v_1, (\tilde{\tau}_1^\theta)(g)v_2] = \chi^{-1}(\theta(g))[v_1, v_2] \tag{4.6}$$

It is easy to see that the above form $[\,,\,]$ is unique up to scalars (hence symmetric or skew-symmetric) and we can attach a sign $\varepsilon_\theta^\chi(\tilde{\tau}_1) \in \{\pm 1\}$ capturing the nature of the form. To simplify notation, we write $\varepsilon_\theta^\chi(\tilde{\tau}_1) = \varepsilon_\theta(\tilde{\tau}_1)$. We can show that the form $[\,,\,]|_{\tau_1 \times \tau_1}$ is non-degenerate and satisfies

$$[\tau_1(g)w_1, \tau_1^\theta(g)w_2] = [w_1, w_2]. \tag{4.7}$$

We refer the reader to lemma 4.14 and lemma 4.15 in [2] for details. It follows from (4.6) and (4.7) that

$$\varepsilon_\theta(\tilde{\tau}_1) = \varepsilon_\theta(\tau_1). \tag{4.8}$$

For $g \in \mathrm{GL}(2,F)$, we let $\alpha(g) = (\mathrm{Int}(x)\circ\theta)(g) = x\theta(g)x^{-1}$. It is clear that $\alpha(g) = 1$ and

$$(\tilde{\tau}_1^{\theta})^{\vee} = (\tilde{\tau}_1^{\alpha})^{\vee} \simeq \tilde{\tau}_1 \otimes \chi.$$

A simple computation shows that

$$\varepsilon_{\alpha}(\tilde{\tau}_1) = \varepsilon_{\theta}(\tilde{\tau}_1)\omega_{\tilde{\tau}_1}(\theta(x)x). \tag{4.9}$$

We refer the reader to §1.2 in [15] for the details. Since $\alpha(g) = 1$ and $\theta(x)x = 1$, it follows that

$$\varepsilon_{\alpha}(\tilde{\tau}_1) = \varepsilon_{\theta}(\tilde{\tau}_1)\omega_{\tilde{\tau}_1}(\theta(x)x) = \varepsilon_{\theta}(\tilde{\tau}_1).$$

It follows from earlier work that $\varepsilon_{\alpha}(\tilde{\tau}_1) = 1$ (see §4.2.5 in [2]) and the result follows.

4.3.2. *π is tempered.* It follows from part (i) of theorem 3.3 above, that there exists a proper parabolic subgroup $P = MN$, and an irreducible smooth discrete series representation σ of M such that $\pi \hookrightarrow \mathrm{ind}_P^G\,\sigma$. Since σ is a discrete series representation, π occurs with multiplicity one in $\mathrm{ind}_P^G\,\sigma$ (see [6], [8]). As observed before, θ satisfies all the hypotheses needed in order to apply Casselman's pairing (see §3 of [15]) and it follows that

$$\varepsilon_{\theta}(\pi) = \varepsilon_{\theta}(\pi_N), \tag{4.10}$$

where π_N is the Jacquet module of π with respect to P.

Let $P = MN$ is a proper parabolic subgroup of G, we have the following possibilities for M. To be precise, we have

(a) $M \simeq \mathrm{GL}(1,F) \times \mathrm{GL}(1,F)$,
(b) $M \simeq \mathrm{GL}(2,F)$,
(c) $M \simeq \mathrm{SL}(2,F) \times \mathrm{GL}(1,F)$.

In all these cases, we show that $\sigma^{\theta} \simeq \sigma^{\vee}$. W record the result in the following lemma.

Lemma 4.4. *Let σ be an irreducible smooth representation of M. Then $\sigma^{\theta} \simeq \sigma^{\vee}$.*

Proof. Consider the mapping $g \to \theta(g)$. For $g \in M$, after making appropriate identifications, it is easy to see that we get the following.

1) If $g \leftrightarrow (a,b)$, $a,b \in \mathrm{GL}(1,F)$ then $\theta(g) \leftrightarrow (a^{-1}, b^{-1})$.

2) If $g \leftrightarrow k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,F)$ then $\theta(g) \leftrightarrow x^{\top}k^{-1}x^{-1},\ x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3) If $g \leftrightarrow (a,h)$, $a \in \mathrm{GL}(1,F)$, $h = \begin{bmatrix} b & c \\ d & e \end{bmatrix} \in \mathrm{SL}(2,F)$ then $\theta(g) \leftrightarrow (a^{-1},\, xhx^{-1})$ where x is as above.

In case (a), the result is clearly true. In cases (b) and (c), the result follows from a simple application of the Gelfand-Kazhdan theorem and Waldspurger's theorem (see theorem 3.2 above) respectively. □

Since σ is irreducible and $\sigma^\theta \simeq \sigma^\vee$, we get a sign $\varepsilon_\theta(\sigma)$. Our goal is to show that $\varepsilon_\theta(\pi) = \varepsilon_\theta(\sigma)$. The key idea is to consider the cases when σ is regular (explained later) and σ is not regular. In the case when σ is regular, we can show that σ occurs with multiplicity one in π_N and hence $\varepsilon_\theta(\pi) = \varepsilon_\theta(\sigma)$. When σ is not regular, we will construct suitably invariant forms on $\mathrm{ind}_P^G(\sigma\nu)$, where $\nu \in X_{nr}^1(M)$ (here $X_{nr}^1(M)$ is the compact torus of unitary unramified characters of M) and deduce the general case from the regular case. Using the fact that σ is an irreducible smooth discrete series Iwahori-spherical representation of M, we can compute the sign $\varepsilon_\theta(\sigma)$.

Lemma 4.5. *Suppose that σ is regular (i.e., $\sigma^n \simeq \sigma$ for $n \in N_G(M)$ implies $n \in M$). Then $\varepsilon_\theta(\pi) = \varepsilon_\theta(\sigma)$.*

Proof. Since σ is regular, $\mathrm{ind}_P^G \sigma$ is irreducible (see Corollary 1.2 in [6]). Therefore, we have $\pi = \mathrm{ind}_P^G \sigma$. The Jacquet module $(\mathrm{ind}_P^G \sigma)_N$ decomposes into generalized eigenspaces under the action of the center of M. As in III.3 in [20], write $(\mathrm{ind}_P^G \sigma)_N^w$ for the sum of those generalized eigenspaces corresponding to unitary eigencharacters. Since σ is regular, $(\mathrm{ind}_P^G \sigma)_N^w$ is semisimple and contains σ with multiplicity one (see III.7.3 in [20]). It follows that $\pi_N = (\mathrm{ind}_P^G \sigma)_N$ also contains σ with multiplicity one. Using corollary in §3.5 in [15], we have $\varepsilon_\theta(\pi) = \varepsilon_\theta(\sigma)$. $\square$

Before we proceed with the general case, we set up some notation and recall a few things we need. Let K be a standard maximal compact subgroup of G such that $G = PK$. We write W for the space of σ and $\mathcal{F}(\sigma)$ for the space of smooth functions $f : K \to W$ such that $f(mnk) = \sigma(m)f(k)$ for $m \in M \cap K, n \in N \cap K$ and $k \in K$. For $\nu \in X_{nr}^1(M)$, we realize $\mathrm{ind}_P^G(\sigma\nu)$ as the space of W valued functions on G in the usual way. It is easy to see that the restriction map

$$f \to f|_K : \mathrm{ind}_P^G(\sigma\nu) \to \mathcal{F}(\sigma)$$

is an isomorphism and we can define a G-action on the space $\mathcal{F}(\sigma)$ via this isomorphism. In this way, we can view the family of representations $\mathrm{ind}_P^G(\sigma\nu)$ as sharing the same underlying space $\mathcal{F}(\sigma)$. In the same way, we can view the family of representations $\mathrm{ind}_P^G((\sigma\nu)^\theta)$ as sharing the same space $\mathcal{F}(\sigma^\theta)$. Let $(\ ,\ )_\sigma$ be a non-degenerate M-invariant form on $W \times W^\theta$, i.e.,

$$(\sigma(m)w_1, \sigma^\theta(m)w_2)_\sigma = (w_1, w_2)_\sigma, \quad m \in M, w_1, w_2 \in W.$$

We fix a Haar measure μ on K and consider the pairing

$$(f, f') \to \int_K (f(k), f'(k))_\sigma \, d\mu(k) : \mathcal{F}(\sigma) \times \mathcal{F}(\sigma^\theta) \to \mathbb{C}.$$

Using the isomorphism

$$\mathrm{ind}_P^G(\sigma\nu) \simeq \mathcal{F}(\sigma), \quad \mathrm{ind}_P^G(\sigma^\theta\nu^{-1}) \simeq \mathcal{F}(\sigma^\theta)$$

we get a non-degenerate G-invariant pairing on $\mathrm{ind}_P^G(\sigma\nu) \times \mathrm{ind}_P^G(\sigma^\theta\nu^{-1})$.

We construct a family of non-degenerate G-invariant pairings on $\mathrm{ind}_P^G(\sigma\nu) \times (\mathrm{ind}_P^G(\sigma\nu))^\theta$ and use it to study the signs $\varepsilon_\theta(\mathrm{ind}_P^G(\sigma\nu))$ (when the induced representation is irreducible. We briefly explain how to construct this family of pairings.

For any function f on G (or K) we define $f^\theta(x) = f(\theta(x))$. It is easy to see that the map

$$f \to f^\theta : (\mathrm{ind}_P^G(\sigma\nu))^\theta \to \mathrm{ind}_P^G(\sigma^\theta\nu^{-1})$$

is a G-isomorphism. As earlier, we view the family of representations $\mathrm{ind}_{\bar{P}}^G(\sigma^\theta\nu^{-1})$ as sharing a common underlying space of smooth functions from K to W which transform under the left action by $\bar{P}\cap K$ according to σ^θ. We denote this space by $\bar{\mathcal{F}}(\sigma^\theta)$. For a dense set $\nu \in X_{nr}^1(M)$, the representations $\mathrm{ind}_{\bar{P}}^G(\sigma^\theta\nu^{-1})$ and $\mathrm{ind}_P^G(\sigma^\theta\nu^{-1})$ are related by standard intertwining operators. For our purposes, it is enough to note that there is a neighborhood $\mathcal{U}$ of the identity in $X_{nr}^1(M)$ and a continuous family of isomorphisms

$$\mathcal{A}(\sigma^\theta,\nu^{-1}) : \mathrm{ind}_{\bar{P}}^G(\sigma^\theta\nu^{-1}) \to \mathrm{ind}_P^G(\sigma^\theta\nu^{-1}), \quad \nu \in \mathcal{U}.$$

Here we interpret continuity as the statement that for any $f \in \mathcal{F}(\sigma)$ and $\bar{f} \in \bar{\mathcal{F}}(\sigma^\theta)$, the map

$$\nu \to (f, \mathcal{A}(\sigma^\theta,\nu^{-1})\bar{f}) : \mathcal{U} \to \mathbb{C}$$

is continuous.

For each $\nu \in \mathcal{U}$, the pairing

$$(f_1,f_2)_\nu = (f_1, \mathcal{A}(\sigma^\theta,\nu^{-1})f_2^\theta), \quad f_1,f_2 \in \mathcal{F}(\sigma)$$

can be viewed as a non-degenerate G-invariant pairing on $\mathrm{ind}_P^G(\sigma\nu) \times (\mathrm{ind}_P^G(\sigma\nu))^\theta$.

Given $f_1 \neq 0$, there is an f_2 such that $(f_1,f_2)_1 \neq 0$. By continuity, $(f_1,f_2)_\nu \neq 0$ in some connected neighborhood $\mathcal{V}$ of the identity in $X_{nr}^1(M)$. Now there is a dense set $\mathcal{D} \subset \mathcal{V}$ such that for $\nu \in \mathcal{D}$, the representation $\mathrm{ind}_P^G(\sigma\nu)$ is irreducible. Hence

$$(f_1,f_2)_\nu = \varepsilon_\theta(\mathrm{ind}_P^G(\sigma\nu))(f_2,f_1)_\nu, \quad \nu \in \mathcal{D}.$$

By continuity, we have signs $\varepsilon_{\theta,\nu}$ for $\nu \in \mathcal{V}$ such that

$$(f_1,f_2)_\nu = \varepsilon_{\theta,\nu}(f_2,f_1)_\nu, \quad \nu \in \mathcal{V},$$

and

$$\varepsilon_{\theta,\nu} = \varepsilon_\theta(\mathrm{ind}_P^G(\sigma\nu)), \quad \nu \in \mathcal{D}.$$

As $\mathcal{V}$ is connected, the map

$$\nu \to \varepsilon_{\theta,\nu} : \mathcal{V} \to \{\pm 1\}$$

must be constant.

Lemma 4.6. $\varepsilon_\theta(\pi) = \varepsilon_\theta(\sigma)$.

Proof. Since $\pi \hookrightarrow \mathrm{ind}_P^G(\sigma)$ with multiplicity one, it follows that the pairing $(\ ,\ )_1$ restricts to a non-zero pairing on the space of π. Therefore, we get

$$\varepsilon_\theta(\pi) = \varepsilon_{\theta,1}.$$

Choose $\nu \in \mathcal{D}$ such that $\sigma\nu$ is regular. It follows from the regular case above (see Lemma 4.5) that

$$\varepsilon_\theta(\mathrm{ind}_P^G(\sigma\nu)) = \varepsilon_{\theta,\nu} = \varepsilon_\theta(\sigma) = \varepsilon_\theta(\sigma\nu).$$

Using the fact that the map $\nu \to \varepsilon_{\theta,\nu} : \mathcal{V} \to \{\pm 1\}$ is constant, it follows that

$$\varepsilon_\theta(\pi) = \varepsilon_\theta(\sigma).$$

$\square$

Since σ is an irreducible discrete series representation of M, it follows that σ is generic. Proceeding as in the previous case and using the fact that σ is Iwahori spherical, we can see that $\varepsilon_\theta(\sigma) = 1$ and hence the result.

Acknowledgements

I would like to thank the referee for his comments on an earlier version of the paper. I would also like to thank professor Alan Roche for his many valuable suggestions, help and encouragement throughout this project.

References

1. Jeffrey D. Adler and Dipendra Prasad, *On certain multiplicity one theorems*, Israel J. Math. **153** (2006), 221–245. MR 2254643
2. Kumar Balasubramanian, *Self-dual representations of* $\mathrm{SL}(n, F)$, Proc. Amer. Math. Soc. **144** (2016), no. 1, 435–444. MR 3415609
3. Armand Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259. MR 0444849 (56 #3196)
4. Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation. MR 1410059 (97i:22005)
5. S. S. Gelbart and A. W. Knapp, *L-indistinguishability and R groups for the special linear group*, Adv. in Math. **43** (1982), no. 2, 101–121. MR 644669 (83j:22009)
6. David Goldberg, *Reducibility of induced representations for* $\mathrm{Sp}(2n)$ *and* $\mathrm{SO}(n)$, Amer. J. Math. **116** (1994), no. 5, 1101–1151. MR 1296726
7. R. Gow, *Real representations of the finite orthogonal and symplectic groups of odd characteristic*, J. Algebra **96** (1985), no. 1, 249–274. MR 808851 (87b:20015)
8. Rebecca A. Herb, *Elliptic representations for* $\mathrm{Sp}(2n)$ *and* $\mathrm{SO}(n)$, Pacific J. Math. **161** (1993), no. 2, 347–358. MR 1242203
9. H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981), no. 2, 199–214. MR 620708
10. Hervé Jacquet, *A correction to conducteur des représentations du groupe linéaire [mr620708]*, Pacific J. Math. **260** (2012), no. 2, 515–525. MR 3001803
11. Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987. MR 1041060 (91f:11040)
12. Dipendra Prasad, *On the self-dual representations of finite groups of Lie type*, J. Algebra **210** (1998), no. 1, 298–310. MR 1656426 (2000a:20025)
13. ______, *On the self-dual representations of a p-adic group*, Internat. Math. Res. Notices (1999), no. 8, 443–452. MR 1687319 (2000d:22019)
14. Dipendra Prasad and Dinakar Ramakrishnan, *Self-dual representations of division algebras and Weil groups: a contrast*, Amer. J. Math. **134** (2012), no. 3, 749–767. MR 2931222
15. Alan Roche and Steven Spallone, *Twisted signs for p-adic linear groups*, preprint.
16. Alan Roche and Ryan Vinroot, *Dualizing involutions for classical and similitude groups over local non-archimedean fields*, preprint.
17. F. Rodier, *Modèle de Whittaker et caractères de représentations*, Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1974), Springer, Berlin, 1975, pp. 151–171. Lecture Notes in Math., Vol. 466. MR 0393355 (52 #14165)
18. Paul J. Sally, Jr. and Marko Tadić, *Induced representations and classifications for* $\mathrm{GSp}(2, F)$ *and* $\mathrm{Sp}(2, F)$, Mém. Soc. Math. France (N.S.) (1993), no. 52, 75–133. MR 1212952
19. Ralf Schmidt, *Iwahori-spherical representations of* $\mathrm{GSp}(4)$ *and Siegel modular forms of degree 2 with square-free level*, J. Math. Soc. Japan **57** (2005), no. 1, 259–293. MR 2114732
20. J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333. MR 1989693

Department of Mathematics,
Indian Institute of Science Education and Research Bhopal,

Bhopal 462066.
E-mail address: bkumar@iiserb.ac.in